Some relationships between the geometry of the tangent bundle and the geometry of the Riemannian base manifold.

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Abstract. We compute the curvature tensor of the tangent bundle of a Riemannian manifold endowed with a natural metric and we get some relationships between the geometry of the base manifold and the geometry of the tangent bundle.

Keywords: Natural tensor fields · Tangent bundle · Riemannian manifolds

Mathematics Subject Clasification (2000): 53C20 · 53B21 · 53A55

1 Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Let $\pi : TM \longrightarrow M$ and $P : O(M) \longrightarrow M$ be the tangent and the orthonormal bundle over M respectively. In this paper we deal with a class of Riemannian metrics G on TM. These metrics makes $\pi : (TM, G) \longrightarrow (M, g)$ a Riemannian submersion, the horizontal distribution induced by the Levi-Civita connection of (M, g) orthogonal to the vertical distribution and G is the image by a natural operator of order two of the metric g. The Sasaki metric and the Cheeger-Gromoll metric are well known examples of these class of metrics, and there were extensively studied by Kowalski [6], Aso [1], Sekizawa [10], Musso and Tricerri [8], Gudmundsson and Kappos [3] among others. The notion of *natural tensor* on the tangent bundle of a Riemannian manifold as a tensor that is the image by a natural operator of order two of the base manifold metric, was introduced and characterized by Kowalski and Sekizawa in [7]. In [2], Calvo and Keilhauer showed that for a given Riemannian manifold (M, g), any $(0, 2)$ tensor field on TM admits a global matrix representation. Using this one to one relationship, they defined and characterized, without making use of the theory of differential invariants, what they also called *natural tensor*. In the symmetric case this concept coincide with the one of Kowalski and Sekizawa. In [4], the first author gives a new approach of the concept of naturality, introducing the notion of *s-space* and *λ-naturality*. This approach avoids jets and natural operators theory and generalized the one given in [2] and [7].

In section 2, we introduce natural metrics on TM by means of [2]. For any $q \in M$, let

M_q be the tangent space of M at q. Let $\psi : N : O(M) \times \mathbb{R}^n \longrightarrow TM$ be the projection defined by

$$\psi(q, u, \xi) = \sum_{i=1}^{n} \xi^i u_i \tag{1}$$

where $u = (u_1, \ldots, u_n)$ is an orthonormal basis for M_q and $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$. It is well known (see [8]), that for a fixed Riemannian metric on TM a suitable Riemannian metric G^* on N can be defined such that $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion. Based on this fact and the O'Neill formula, in Section 3, we compute the curvature tensor of (TM, G), when G is a natural metric. As an application, we get in Section 4 some relationships between the geometry of TM and the geometry of M.

Throughout, all geometric objets are assumed to be differentiable, i.e. C^∞.

2 Preliminaries.

Let ∇ be the Levi-Civita connection of g and $K : TTM \longrightarrow TM$ the connection map induced by ∇. For any $q \in M$ and $v \in M_q$, let $\pi_{*_v} : (TM)_v \longrightarrow M_q$ be the differential map of π at v, and $K_v : (TM)_v \longrightarrow M_q$ the restriction of K to $(TM)_v$.

Since the linear map $\pi_{*_v} \times K_v : (TM)_v \longrightarrow M_q \times M_q$ defined by $(\pi_{*_v} \times K_v)(b) = (\pi_{*_v}(b), K_v(b))$ is an isomorphism that maps the horizontal subspace $(TM)^h_v = \ker K_v$ onto $M_q \times \{0_q\}$ and the vertical subspace $(TM)^v_v = \ker \pi_{*_v}$ onto $\{0_q\} \times M_q$, where 0_q denotes the zero vector, we define differentiable mappings $e_i, e_{n+i} : N = O(M) \times \mathbb{R}^n \longrightarrow TTM$ for $i = 1, \ldots, n$ and $v = \psi(q, u, \xi)$ by

$$\begin{aligned} e_i(q, u, \xi) &= (\pi_{*_v} \times K_v)^{-1}(u_i, 0_q) \\ e_{n+i}(q, u, \xi) &= (\pi_{*_v} \times K_v)^{-1}(0_q, u_i) \end{aligned} \tag{2}$$

The action of the orthonormal group $O(n)$ of $\mathbb{R}^{n \times n}$ on N is given by the family of maps $R_a : N \longrightarrow N$, $a \in O(n)$, $R_a(q, u, \xi) = (q, u.a, \xi.a)$ where $u.a = (\sum_{i=1}^n a^i_1 u_i, \ldots, \sum_{i=1}^n a^i_n u_i)$ and $\xi.a = (\sum_{i=1}^n a^i_1 \xi^i, \ldots, \sum_{i=1}^n a^i_n \xi^i)$. It is easy to see that

$$\{e_i(R_a(p, u, \xi))\} = \{e_l(p, u, \xi)\}.L(a)$$

where $L : O(n) \longrightarrow \mathbb{R}^{2n \times 2n}$ is the map defined by

$$L(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{3}$$

For any $(0, 2)$ tensor field T on TM we define the differentiable function ${}^gT : N \longrightarrow \mathbb{R}^{2n \times 2n}$ as follows: If $(q, u, \xi) \in N$ and $v = \psi(q, u, \xi)$, let ${}^gT(q, u, \xi)$ be the matrix of the bilinear form $T_v : (TM)_v \times (TM)_v \longrightarrow \mathbb{R}$ induced by T on $(TM)_v$ with respect to the

basis $\{e_1(q,u,\xi),\dots,e_{2n}(q,u,\xi)\}$. One sees easily that ${}^gT$ satisfies the following invariance property:

$$ {}^gT \circ R_a = (L(a))^t.{}^gT.L(a) \tag{4} $$

Moreover, there is a one to one correspondence between the $(0,2)$ tensor fields on TM and differentiable maps ${}^gT$ satisfying (4).

A tensor field T on TM will be call natural with respect to g if ${}^gT$ depends only of the parameter ξ, (see [2]). In the sense of [4], the collection $\lambda = (N, \psi, O(n), \tilde{R}, \{e_i\})$ is a s-space over TM, with base change morphism L; and the natural tensors with respect to g are the $\lambda - natural$ tensors with respect to TM.

In this paper we will call G a natural metric on TM if:

1. G is a Riemannian metric such that $\pi : (TM, G) \longrightarrow (M, g)$ is a Riemannian submersion.
2. For $v \in TM$, the subspaces $(TM)^v_v$ and $(TM)^h_v$ are orthogonals.
3. G is natural with respect to g.

From Lemma 3.1 of [2], it follows that G is a natural metric on TM if

$$ {}^gG(p,u,\xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & \alpha(\|\xi\|^2).Id_{n\times n} + \beta(\|\xi\|^2)(\xi)^t.\xi \end{pmatrix} \tag{5} $$

where $\alpha, \beta : [0, +\infty) \longrightarrow \mathbb{R}$ are differentiable functions satisfying $\alpha(t) > 0$, and $\alpha(t) + t\beta(t) > 0$ for all $t \geq 0$.

Remark 2.1 *The Sasaki metric G_s corresponds to the case $\alpha = 1,\ \beta = 0$; and the Cheeger-Gromoll metric G_{ch} to the case $\alpha = \beta$, and $\alpha(t) = \frac{1}{1+t}$.*

3 Curvature equations.

In this section we compute the curvature tensor of TM endowed with a natural metric. Since this computation involves well known objects defined on N, we shall begin to describe them briefly using the connection map.

3.1 Canonical constructions on N.

Let θ^i, ω^i_j be the canonical 1-forms on $O(M)$, which in terms of the connection map are defined as follows:

$$ \theta^i(q,u)(b) = g_q\Big(P_{*_{(q,u)}}(b), u_i\Big) \tag{6} $$

$$ \omega^i_j(q,u)(b) = g_q\Big(K((\pi_j)_{*_{(q,u)}}(b)), u_i\Big) \tag{7} $$

where $\pi_j : O(M) \longrightarrow TM$ is the j^{th} projection, i.e. $\pi_j(q,u) = u_j$ and $1 \leq i,j \leq n$.

From now on, let θ^i, ω^i_j, $d\xi^i$ be the pull backs of the canonical 1-forms and the usual 1-forms on $\mathbb{R}^n$ by $P_1 : N \longrightarrow O(M)$ and $P_2 : N \longrightarrow \mathbb{R}^n$.

For any $z \in N$ let us denote by $V_z = \ker \psi_{*_z}$ and $H_z = \{b \in N_z : \omega^i_j(z)(b) = 0, 1 \leq i < j \leq n\}$ the vertical and the horizontal subspace of N_z respectively. By letting [8]

$$\theta^{n+i} = d\xi^i + \sum_{j=1}^{n} \xi^j.\omega^i_j \tag{8}$$

we get that for any $z \in N$, $\{\theta^1(z), \ldots, \theta^{2n}(z), \{\omega^i_j(z)\}\}$ is a basis for N^*_z and $V_z = \{b \in N_z : \theta^l(z)(b) = 0 \text{ for } 1 \leq l \leq 2n\}$.

Let $H_1, \ldots, H_{2n}, \{V^l_m\}_{1\leq l<m\leq n}$ be the dual frame of $\{\theta^1, \ldots, \theta^{2n}, \{\omega^i_j\}\}$. The vector fields were constructed as follows: If $z = (q,u,\xi)$, let c_i be the geodesic that satisfies $c_i(0) = q$ and $\dot{c}_i(0) = u_i$. Let $E^i_1, \ldots, E^i_n$ be the parallel vector fields along c_i such that $E^i_l(0) = u_l$. If we define $\gamma_i(t) = (c_i(t), E^i_1(t), \ldots, E^i_n(t), \xi)$, then

$$H_i(z) = \dot{\gamma}_i(z) \tag{9}$$

$$H_{n+i}(z) = (i_{(q,u)})_{*_\xi}(\frac{\partial}{\partial \xi^i}|_\xi) \tag{10}$$

for $1 \leq i \leq n$, where $i_{(q,u)} : \mathbb{R}^n \longrightarrow N$ is the inclusion map given by $i_{(q,u)}(\xi) = (q,u,\xi)$.

Let $\sigma_z : O(n) \longrightarrow N$ be the map defined by $\sigma_z(a) = R_a(z) = z.a$. Since $V_z = \ker(\psi_{*_z}) = (\sigma_z)_{*_{Id}}(\mathfrak{o}(n))$, where $\mathfrak{o}$ is the space of skew symmetric matrices of $R^{n\times n}$, let

$$V^l_m(z) = (\sigma_z)_{*_{id}}(A^l_m) \tag{11}$$

where $[A^l_m]^l_m = 1$, $[A^l_m]^m_l = -1$ and $[A^l_m]^i_j = 0$ otherwise. Hence,

$$\psi_{*_z}(V^l_m(z)) = 0 \tag{12}$$

An easy check shows that

$$\psi_{*_z}(H_i(z)) = e_i(z) \tag{13}$$

and

$$\psi_{*_z}(H_{n+i}(z)) = e_{n+i}(z) \tag{14}$$

Let $\omega = \sum_{1\leq i<j\leq n} \omega^i_j \otimes \omega^i_j$, if G is a Riemannian metric on TM then

$$G^* = \psi^*(G) + \omega \tag{15}$$

is also a Riemannian metric on N. It follows easily that $(V_z) \perp_{G^*} H_z$ and $\psi_{*_z} : H_z \longrightarrow (TM)_{\psi(z)}$ is an isometry, therefore $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion. We shall use this fact to compute the curvature tensor of (TM, G) when G is a natural metric.

Remark 3.1 *Let X be a vector field on TM, the horizontal lift of X is a vector field X^h on N such that $X^h(z) \in H_z$ and $\psi_{*_z}(X^h(z)) = X(\psi(z))$. If $X(\psi(z)) = \sum_{i=1}^{2n} x^i(z)e_i(z)$, from (12), (13) and (14) it follows that $X^h(z) = \sum_{i=1}^{2n} x^i(z)H_i(z)$.*

Proposition 3.2 *For $1 \leq i,j,l,m \leq n$ let $R_{ijlm} : N \longrightarrow \mathbb{R}$ be the maps defined by $R_{ijlm}(q,u,\xi) = g(R(u_i,u_j)u_l,u_m)$, where R is the curvature tensor of (M,g). The Lie bracket on vertical and horizontal vector field on N satisfies:*

a) $[H_i,H_j] = \sum_{l,m=1}^n R_{ijlm}\xi^m H_{n+l} + \frac{1}{2}\sum_{l,m=1}^n R_{ijlm}V_m^l$.

b) $[H_i,H_{n+j}] = 0$.

c) $[H_i,V_m^l] = \delta_{il}H_m - \delta_{im}H_l$.

d) $[H_{n+i},H_{n+j}] = 0$.

e) $[H_{n+i},V_m^l] = \delta_{il}H_{n+m} - \delta_{im}H_{n+l}$.

f) $[V_j^i,V_m^l] = \delta_{il}V_{mj} + \delta_{jl}V_{im} + \delta_{im}V_{jl} + \delta_{jm}V_{li}$.

g) If $f : N \longrightarrow \mathbb{R}$ is a function that depends only on the parameter ξ, then $H_i(f) = 0$ and $V_j^i(f) = \xi^i H_{n+j}(f) - \xi^j H_{n+i}(f)$.

h) If $X,Y \in \chi(TM)$ and $v = \psi(q,u,\xi)$ then $[X^h,Y^h]^v|_{(q,u,\xi)} = \sum_{1\leq l<m\leq n} g_q(R(\pi_*(X(v)),\pi_*(Y(v)))u_l,u_m)V_m^l(q,u,\xi)$.

The proof is straightforward and follows by taking local coordinates in M and the induced one in TM and evaluating the forms θ^i, θ^{n+i}, ω_j^i on the fields $[H_r,H_s]$, $[H_r,V_m^l]$ and $[V_m^l,V_{m'}^{l'}]$ for $1 \leq r,s \leq 2n$, $1 \leq l < m \leq n$ and $1 \leq l' < m' \leq n$.

3.2 The main result.

From now on, let $\bar{R}$ and R^* be the curvature tensors of (TM,G) and (N,G^*). For simplicity we denote by $<\ ,\ >$ the metrics G and G^*. Since $\psi : (N,G^*) \longrightarrow (TM,G)$ is a Riemannian submersion, by the O'Neill formula (see [9]) we have that

$$\begin{aligned} <\bar{R}(X,Y)Z,W> \circ\, \psi \quad &= <R^*(X^h,Y^h)Z^h,W^h> + \tfrac{1}{4} <[Y^h,Z^h]^v,[X^h,W^h]^v> \\ &-\tfrac{1}{4}<[X^h,Z^h]^v,[Y^h,W^h]^v> - \tfrac{1}{2}<[Z^h,W^h]^v,[X^h,Y^h]^v> \end{aligned} \tag{16}$$

If $Y^h(z) = \sum_{i=1}^{2n} y^j(z)H_i(z)$, $Z^h(z) = \sum_{i=1}^{2n} z^k(z)H_i(z)$ and $W^h(z) = \sum_{i=1}^{2n} w^l(z)H_i(z)$, then the first term of the right side of equality (16) is

$$<R^*(X^h,Y^h)Z^h,W^h> = \sum_{ijkl=1}^{2n} x^i y^j z^k w^l <R^*(H_i,H_j)H_k,H_l>$$

On the other hand, if $v = \psi(q,u,\xi)$, it follows from Proposition 3.2 (part h) that

$$< [X^h, Y^h]^v, [Z^h, W^h]^v > |_{(q,u,\xi)} =$$

$$= \frac{1}{2}\sum_{r,s=1}^{n} < R(\pi_*(X(v)), \pi_*(Y(v)))u_r, u_s > . < R(\pi_*(Z(v)), \pi_*(W(v)))u_r, u_s > \qquad (17)$$

Remark 3.3 *In order to compute* $< \bar{R}(X(v), Y(v))Z(v), W(v) >$ *it is sufficient to evaluate the right side of (16) on points of* N *of the form* $z = (q, u, t, 0, \dots, 0)$ *such that* $v = \psi(z) = t.v$ *and* $t = \|v\|$.

Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ be a differentiable map, from now on, let us denote by $\dot{f}(t)$ the derivate of f at t.

Theorem 3.4 *Let* G *be a natural metric on* TM*, and* α, β *be the functions that characterizes* G*. If* $1 \le i, j, k, l \le n$ *and* $z = (q, u, t, 0, \dots, 0)$ *we have that*

a) $< R^*(H_i(z), H_j(z))H_k(z), H_l(z)) >=$

$$t^2\alpha(t^2).\sum_{r=1}^{n}\Big\{\frac{1}{2}R_{ijr1}(z)R_{klr1}(z) + \frac{1}{4}R_{ilr1}(z)R_{kjr1}(z) + \frac{1}{4}R_{jlr1}(z)R_{ikr1}(z)\Big\}$$

$$+ \sum_{1\le r<s\le n}\Big\{\frac{1}{2}R_{ijr1}(z)R_{klrs}(z) + \frac{1}{4}R_{ilr1}(z)R_{kjrs}(z) + \frac{1}{4}R_{jlr1}(z)R_{ikrs}(z)\ \Big\} + R_{ijkl}(z).$$

b) Let $\epsilon_{ijkl} = \delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}$*, then*

b.1) If no index is equal to one, then

$$< R^*(H_{n+i}(z), H_{n+j}(z))H_{n+k}(z), H_{n+l}(z) >= \epsilon_{ijkl}F(t^2)$$

where $F : [0, +\infty) \longrightarrow \mathbb{R}$ *is defined by*

$$F(t) = \frac{\alpha(t)\beta(t) - t(\dot{\alpha}(t))^2 - 2\alpha(t)\dot{\alpha}(t)}{\alpha(t) + t\beta(t)} \qquad (18)$$

b.2) If some index equals one, for example $l = 1$*, then*

$$< R^*(H_{n+i}(z), H_{n+j}(z))H_{n+k}(z), H_{n+1}(z) >= \epsilon_{ijk1}H(t^2)$$

where $H : [0, +\infty) \longrightarrow \mathbb{R}$ *is defined by*

$$H(t) = \phi(t)\frac{\partial}{\partial t}\ln(\alpha\Delta)|_t - 2\dot{\phi}(t) \qquad (19)$$

and $\phi(t) = \alpha(t) + t\dot{\alpha}(t)$*,* $\Delta(t) = \alpha(t) + t\beta(t)$*.*

c) $< R^*(H_i(z), H_{n+j}(z))H_{n+k}(z), H_{n+l}(z) >= 0.$

d) $< R^*(H_{n+i}(z), H_{n+j}(z))H_k(z), H_l(z) >=$

$$= \frac{1}{2}(2\alpha(t^2) + (\delta_{i1} + \delta_{j1})\beta(t^2)t^2)R_{ijkl}(z) + \frac{1}{2}\delta_{i1}(\beta(t^2) - 2\dot{\alpha}(t^2))t^2 R_{klj1}(z)$$

$$+\frac{1}{2}\delta_{j1}(2\dot{\alpha}(t^2) - \beta(t^2))t^2 R_{kli1}(z) + \frac{(\alpha(t^2))^2 t^2}{4}\sum_{r=1}^{n}\{R_{krj1}(z)R_{rli1}(z) - R_{kri1}(z)R_{rlj1}(z)\}.$$

e) $< R^*(H_i(z), H_{n+j}(z))H_k(z), H_{n+l}(z) >=$

$$\frac{1}{2}\alpha(t^2)R_{kilj}(z) + \frac{(\alpha(t^2))^2 t^2}{4}\sum_{r=1}^{n} R_{krj1}(z)R_{ril1}(z) + \frac{t^2}{2}(\delta_{j1} + \delta_{l1})\dot{\alpha}(t^2)(R_{kil1}(z) - R_{kij1}(z)).$$

f) $< R^*(H_i(z), H_j(z))H_{n+k}(z), H_l(z)) >=$

$$\frac{\alpha(t^2)t}{2}\{< \nabla_D R(E_j^i(s), E_j^l(s))E_j^k(s)|_{s=0}, u_1 > - < \nabla_D R(E_i^j(s), E_i^l(s))E_i^k(s)|_{s=0}, u_1 >\}.$$

The proof follows from the Koszul formula and Proposition 3.2 and it involves a lot of calculation. For more details we refer the reader to [5] pages 132-151.

Theorem 3.5 *The curvature tensor $\bar{R}$ evaluated on $e_i(z)$, $e_{n+i}(z)$ satisfies:*

a) $< \bar{R}(e_i(z), e_j(z))e_k(z), e_l(z) >=$

$$t^2\alpha(t^2)\sum_{r=1}^{n}\{\frac{1}{2}R_{ijr1}(z)R_{klr1}(z) + \frac{1}{4}R_{ilr1}(z)R_{kjr1}(z) + \frac{1}{4}R_{jlr1}(z)R_{ikr1}(z)\} \; + R_{ijkl}(z).$$

b) b.1) If no index is equal to one, then

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) >= \epsilon_{ijkl}.F(t^2) \tag{20}$$

b.2) If some index equals one, for example $l = 1$, then

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+1}(z) >= \epsilon_{ijk1}.H(t^2) \tag{21}$$

c) $< \bar{R}(e_i(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) >= 0.$

d) $< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_k(z), e_l(z) >=$

$$\frac{1}{2}\Big(2\alpha(t^2) + (\delta_{i1} + \delta_{j1})\beta(t^2)t^2\Big)R_{ijkl}(z) + \frac{1}{2}\delta_{i1}\Big(\beta(t^2) - 2\dot{\alpha}(t^2)\Big)t^2 R_{klj1}(z)$$

$$+\frac{1}{2}\delta_{j1}\Big(2\dot{\alpha}(t^2) - \beta(t^2)\Big)t^2 R_{kli1}(z) + \frac{(\alpha(t^2))^2 t^2}{4}\sum_{r=1}^{n}\{R_{krj1}(z)R_{rli1}(z) - R_{kri1}(z)R_{rlj1}(z)\}$$

e) $< \bar{R}(e_i(z), e_{n+j}(z))e_k(z), e_{n+l}(z) >=$

$$\frac{1}{2}\alpha(t^2)R_{kilj}(z)+\frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}R_{krj1}(z)R_{ril1}(z)+\frac{t^2}{2}(\delta_{j1}+\delta_{l1})\dot{\alpha}(t^2)(R_{kil1}(z)-R_{kij1}(z))$$

f) $< \bar{R}(e_i(z), e_j(z))e_{n+k}(z), e_l(z)) >=$

$$\frac{\alpha(t^2)t}{2}\{< \nabla_D R(E^i_j(s), E^l_j(s))E^k_j(s)|_{s=0}, u_1 > - < \nabla_D R(E^j_i(s), E^l_i(s))E^k_i(s)|_{s=0}, u_1 >\}$$

Proof. The proof is straightforward and follows form Theorem 3.4 and equality (16). □

The functions F and H satisfy the following Proposition

Proposition 3.6 *Let $\alpha, \beta : [0, +\infty) \longrightarrow \mathbb{R}$ be differentiable functions such that $\alpha(t) > 0$ and $\alpha(t) + t\beta(t) > 0$ for all $t \geq 0$. If F is the zero function, then:*

i) $\beta(t) = \frac{t(\dot{\alpha}(t))^2+2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$.

ii) $\alpha(t)(\alpha(t) + t\beta(t)) = (t\dot{\alpha}(t) + \alpha(t))^2$.

iii) $\alpha(t) + t\dot{\alpha}(t) > 0$.

iv) $H(t) = 0$ *for all* $t \geq 0$.

Proof. Assertion i) follows from equality (18) and ii) is a consequence of i). Equality ii) shows that $\alpha(t) + t\dot{\alpha}(t) \neq 0$ for all $t \geq 0$, and since $\alpha(0) + 0.\dot{\alpha}(0) = \alpha(0) > 0$, then we get iii). Equality ii) says that $\alpha.\Delta = \phi^2$, and assertion iii) says that $\phi > 0$. Therefore, from equality (19) we get that $H = 0$. □

Corollary 3.7 *Let $\alpha, \beta : [0, +\infty) \longrightarrow \mathbb{R}$ be differentiable functions such that $\alpha(t) > 0$, $\alpha(t) + t\dot{\alpha}(t) > 0$ and $\alpha(t) + t\beta(t) > 0$ if $t \geq 0$. If H is the zero function, then it is also F.*

Proof. Since $\phi > 0$ and $H = 0$, the equality (19) implies that $\ln(\alpha\Delta) = \ln(\phi^2) + C$ for some constant C. In particular $2\ln(\alpha(0)) = 2\ln(\alpha(0)) + C$, hence $C = 0$. Since $\alpha.\Delta = \phi^2$, we obtain that $F = 0$.

□

4 Geometric consequences of curvature equations.

In this section the Riemannian metric G on TM is assumed natural. As trough all the paper, G is characterized by the functions α and β. As in Remark 3.3, if $v \in TM$, let $z = (q, u, t, 0, \ldots, 0) \in N$ such that $\psi(z) = v$ and $t = \|v\|$. From Theorem 3.5 and Proposition 3.6 we get inmediatly

Corollary 4.1 *If (TM, G) is flat then (M, G) is flat.*

Proof. It follows from part a) of Theorem 3.5 by setting $t = 0$. $\square$

Corollary 4.2 *If $\dim M \geq 3$, (TM, G) is flat if and only if (M, g) is flat and*

$$\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$$

Proof. Assume that (TM, G) is flat. From Theorem 3.5 part b.1) and $1 < i < j \leq n$ we have that

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+i}(z), e_{n+j}(z) >= -F(t^2)$$

Therefore $F = 0$, and the desired equality on β follows from Proposition 3.6 part i).

Assuming that (M, g) is flat and $\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$, we only need to show that

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) >= 0 \tag{22}$$

for $1 \leq i, j, k, l \leq 2n$. The other cases also satisfies (22) because $R = 0$. Equality on β implies that $F = 0$, therefore by Proposition 3.6 part iv) we have that $H = 0$, and equality (22) is satisfied. $\square$

We have also immediately the following result

Corollary 4.3 *If $\dim M = 2$, (TM, G) is flat if and only if (M, g) is flat and $H = 0$.*

Remark 4.4 *Let $\alpha(t) > 0$ be a differentiable function that satisfies $t\dot{\alpha}(t) + \alpha(t) > 0$ for all $t \geq 0$ and define $\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$. If we consider the natural metric G induced by α and β, then (TM, G) is flat if (M, g) is flat.*

Remark 4.5 *The above Corollaries generalizes the well known fact that (TM, G_s) is flat if and only if (M, g) if flat (Kowalski [6], Aso [1]). This fact, follows from the Corollaries taking $\alpha = 1$ and $\beta = 0$.*

We will denote by K and $\bar{K}$ the sectional curvatures of (M, g) and (TM, G) respectively.

Theorem 4.6 *We have the following expression for the sectional curvature of (TM, G), where $z = (q, u, t, 0, \ldots, 0)$ and $\psi(z) = v$ with $t = \|v\|$:*

a) For $1 \leq i, j \leq n$:

$$\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2$$

b) b.1) If $2 \leq i, j \leq n$ *and* $i \neq j$

$$\bar{K}(e_{n+i}(z), e_{n+j}(z)) = \frac{F(t^2)}{(\alpha(t^2))^2}$$

b.2) If $2 \leq i \leq n$

$$\bar{K}(e_{n+1}(z), e_{n+j}(z)) = \frac{H(t^2)}{\alpha(t^2)(\alpha(t^2) + t^2\beta(t^2))}$$

c) For $1 \leq i, j \leq n$*:*

$$\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(t^2)}{4}|R(u_j, v)u_i|^2$$

In particular $\bar{K}(e_i, e_{n+1}) = 0$ if $1 \leq i \leq n$, since $v = tu_1$.

Proof. From equality (5) we get that $e_1(z), \ldots, e_{2n}(z)$ is an orthogonal basis for $(TM)_v$ such that $< e_i(z), e_j(z) >= \delta_{ij}$ if $1 \leq i, j \leq n$, $< e_{n+1}(z), e_{n+1}(z) >= \alpha(t^2) + t^2\beta(t^2)$ and $< e_{n+i}(z), e_{n+i}(z) >= \alpha(t^2)$ if $2 \leq i \leq n$. Let $1 \leq i, j \leq n$, $i \neq j$. By setting $k = j$ and $l = i$ in equation a) of Theorem 3.5 we have that

$$\bar{K}(e_i(z), e_j(z)) = - < \bar{R}(e_i(z), e_j(z))e_j(z), e_i(z) >= R_{ijji}(z) - \frac{3}{4}t^2\alpha(t^2)\sum_{r=1}^{n} R_{ij1r}^2(z)$$

Since $K(u_i, u_j) = R_{ijji}(z)$ and $v = tu_1$, we can write

$$\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2$$

Part b) follows directly from equations b.1) and b.2) of Theorem 3.5.

Since $\|e_i(z)\| = 1$ and $< e_i(z), e_{n+j}(z) >= 0$ for $1 \leq i, j \leq n$, from Theorem 3.5 equation e), we see that

$$\bar{K}(e_i(z), e_{n+j}(z)) = -\frac{(\alpha(|v|^2))^2|v|^2}{4(\alpha(|v|^2) + \delta_{j1}\beta(|v|^2)|v|^2)}\sum_{r=1}^{n} R_{irj1}(z)R_{rij1}(z)$$

$$= \frac{\alpha(|v|^2)}{4}\sum_{r=1}^{n}\Big[g(R(u_j, u_1|v|)u_i, u_r)\Big]^2 = \frac{\alpha(|v|^2)}{4}|R(u_j, v)u_i|^2.$$

□

Corollary 4.7

i) (TM, G) *is never a manifold with negative sectional curvature.*

ii) If $\bar{K}$ *is constant, then* (TM, G) *and* (M, g) *are flat.*

iii) If $\bar{K}$ *is bounded and* $\lim_{t \to +\infty} t\alpha(t) = +\infty$*, then* (M, g) *is flat.*

iv) If $c \leq \bar{K} \leq C$ (possibly $c = -\infty$ and $C = +\infty$), then $c \leq K \leq C$.

Proof. Assertions i), ii) and ii) follow from Theorem 4.6 part c). Let $q \in M$ and $u = (u_1, \ldots, u_n)$ be an orthonormal basis for M_q. Then, if we consider $z = (q, u, 0, \ldots, 0)$ and $v = 0_q$, from Theorem 4.6 part a) we have that $\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j)$ and part iv) holds. Also ii) follows from Theorem 3.5) part a) taking $t = 0$.

□

Corollary 4.8 *Let (M, g) be a manifold of constant sectional curvature K_0 and TM endowed with a natural metric G, then we have for $z = (q, u, t, 0, \ldots, 0)$ and $\psi(z) = v$ that*

a) $\bar{K}(e_i(z), e_j(z)) = K_0 - \frac{3}{4}(K_0)^2\alpha(|v|^2)(\delta_{i1} + \delta_{j1})|v|^2$ with $i \neq j$.

b) $\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(|v|^2)}{4} K_0|v|^2(\delta_{ij} + \delta_{i1})$.

The vertical case $\bar{K}(e_{n+i}, e_{n+j})$ is as Theorem 4.6 part b).

From Theorem 4.6 we get the following result

Corollary 4.9 *Let G_1 and G_2 be two natural metrics on TM such that are characterized by the functions $\{\alpha_i\}_{i=1,2}$ and $\{\beta_i\}_{i=1,2}$. If $\bar{K}_1(u)(V, W) = \bar{K}_2(u)(V, W)$ for all $u \in TM$ and $V, W \in (TM)_u$ and (M, g) is not flat, then $\alpha_1 = \alpha_2$.*

Remark 4.10 *Let $G_{+\exp}$ and $G_{-\exp}$ be the natural metrics on TM defined by*

$$^gG_{+\exp}(q, u, \xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & A^+(\xi) \end{pmatrix} \quad \textit{and} \quad ^gG_{-\exp}(q, u, \xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & A^-(\xi) \end{pmatrix}$$

where $A^+(\xi) = e^{|\xi|^2}(Id_{n\times n} + \xi^t.\xi)$ and $A^-(\xi) = e^{-|\xi|^2}(Id_{n\times n} + \xi^t.\xi)$. We call $G_{+\exp}$ and $G_{-\exp}$ the positive and negative exponential metric.

It is known ([10]) that TM endowed with the Cheeger-Gromoll metric is never a manifold of constant sectional curvature. Theorem 4.6 applied to $G_{+\exp}$ and $G_{-\exp}$ shows that these metrics satisfy the same property.

4.1 Ricci tensor and scalar curvature.

Let $Ricc$ and $\bar{R}icc$ be the Ricci tensor of (M, g) and (TM, G) respectively. We will denote by S and $\bar{S}$ the scalar curvature of (M, g) and (TM, G).

Theorem 4.11 *For $1 \leq i, j \leq n$ and $z = (q, u, t, 0 \ldots, 0)$ we have the following expressions for $\bar{R}icc$:*

a) $\bar{R}icc(e_i(z), e_j(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1 \le r,l \le n} R_{irl1}(z) R_{jrl1}(z) + Ricc(u_i, u_j)$

b) $\bar{R}icc(e_i(z), e_{n+j}(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1 \le r \le n} \Big\{ < \nabla_D R(E^i_r, E^r_r) E^j_r|_{s=0}, u_1 >$

$$- < \nabla_D R(E^r_i, E^r_i) E^j_i|_{s=0}, u_1 > \Big\}$$

c) c.1) If $2 \le i \le n$*, then*

$$\bar{R}icc(e_{n+i}(z), e_{n+i}(z)) = \frac{t^2\alpha(t^2)}{4} \sum_{1 \le r,l \le n} R^2_{rli1}(z) + \frac{(n-2)}{\alpha(t^2)} F(t^2) + \frac{1}{\alpha(t^2) + t^2\beta(t^2)} H(t^2)$$

c.2) If $2 \le i, j \le n$ *and* $i \neq j$*, then*

$$\bar{R}icc(e_{n+i}(z), e_{n+j}(z)) = \frac{t^2\alpha(t^2)}{4} \sum_{1 \le r,l \le n} R_{rli1}(z) R_{rlj1}(z)$$

c.3) If $1 \le j \le n$*, then*

$$\bar{R}icc(e_{n+1}(z), e_{n+j}(z)) = \frac{(n-1)}{\alpha(t^2)} H(t^2) \delta_{j1}$$

Proof. Let $\bar{e}_1(z), \ldots, \bar{e}_{2n}(z)$ be the orthonormal basis for $(TM)_v$ induced by the orthogonal basis $e_1(z), \ldots, e_{2n}(z)$, where $\psi(z) = v$. For $X, Y \in (TM)_v$ we have that

$$\bar{R}icc(X, Y) = \sum_{l=1}^{2n} < \bar{R}(X, \bar{e}_l(z))\bar{e}_l(z), Y >$$

Equalities a), b) and c) follow directly from Theorem 3.5 and the fact that $< e_{n+1}(z), e_{n+1}(z) >= \alpha(t^2) + t^2\beta(t^2)$ and $< e_{n+i}(z), e_{n+i}(z) >= \alpha(t^2)$ if $2 \le i \le n$. □

Corollary 4.12 *Let* α *and* β *be the functions that characterizes* G*, such that* $\alpha(t) + t\dot{\alpha}(t) > 0$ *for* $t \ge 0$*. If* (TM, G) *is Ricci flat then* (M, g) *and* (TM, G) *are flats.*

Proof. In order to prove that $R = 0$, it is enough to show that for any $q \in M$ and any orthonormal basis $u = \{u_1, \ldots, u_n\}$ for M_q the following equalities are satisfied

$$< R(u_r, u_l)u_i, u_1 >= 0 \tag{23}$$

for $1 \le r, l \le n$ and $2 \le i \le n$. Let $v \in M_q$, $v \neq 0$ and $z = (q, u, t, 0, \ldots, 0) \in N$ such that $\psi(z) = tu_1 = v$. If $\bar{R}icc = 0$, from Theorem 4.11 part c.3) we have that $H = 0$. Since $\alpha(t) + t\dot{\alpha}(t) > 0$, we get from Corollary 3.7 that $F = 0$. Consequently, equalities (23) follows from c.1). Since $R = 0$ and $H = F = 0$, from Theorem 3.5 we have that $\bar{R} = 0$.

□

Remark 4.13 *Is easy to see from Theorem 4.11 that if* (M, g) *is not flat or if not exists a constant* k *such that* $H(t) = k\alpha(t)$ *and* $(n-2)[\alpha(t) + t\beta(t)]F(t) = \alpha(t)k\Big[(n-2)\alpha(t) + (n-1)t\beta(t)\Big]$, *then* $\bar{R}icc$ *is not a* $\lambda-$*natural tensor (see [4]).*

Corollary 4.14 *Let* $v \in TM$ *and* $z = (\pi(v), u_1, \dots, u_n, t, 0 \dots, 0) \in N$ *such that* $v = u_1 t$. *The scalar curvature of* (TM, G) *at* v *is given by*

$$\bar{S}(v) = S(\pi(v)) - \frac{t^2\alpha(t^2)}{4}\sum_{irl=1}^{n} R^2_{irl1}(z) + \frac{2(n-1)}{\alpha(t^2)\Big(\alpha(t^2) + \beta(t^2)t^2\Big)}H(t^2)$$

$$+\frac{(n-1)(n-2)}{(\alpha(t^2))^2}F(t^2)$$

Proof. Since $\{\bar{e}_1(z), \dots, \bar{e}_{2n}(z)\}$ is an orthonormal basis for $(TM)_v$ and the scalar curvature $\bar{S}(v) = \sum_{l=1}^{2n} Ricc(\bar{e}_l(z), \bar{e}_l(z))$, the expression for $\bar{S}$ follows straightforward from Theorem 4.11.

□

Remark 4.15 *Corollary 4.14 applied to* $G_{+\exp}$ *and* $G_{-\exp}$ *reads:*

$$S_{+\exp}(v) = S(\pi(v)) - (n-1)e^{-|v|^2}\frac{\Big[2 + (n-2)(1+|v|^2)\Big]}{(1+|v|^2)}$$

$$-\frac{e^{|v|^2}}{4}\sum_{i,j=1}^{n}|R(u_i, u_j)v|^2$$

$$S_{-\exp}(v) = S(\pi(v)) + \frac{(n-1)e^{|v|^2}}{1+|v|^2}\Big[(n-2)(3-|v|^2) + \frac{6+2|v|^2}{1+|v|^2}\Big]$$

$$-\frac{e^{-|v|^2}}{4}\sum_{i,j=1}^{n}|R(u_i, u_j)v|^2$$

Proposition 4.16 *If* (M, g) *is a manifold of constant sectional curvature* K_0, *then*

$$S_{+\exp}(v) = (n-1)\Big\{K_0\Big(n - \frac{K_0}{2}|v|^2 e^{|v|^2}\Big) - e^{-|v|^2}\frac{\Big[2 + (n-2)(1+|v|^2)\Big]}{(1+|v|^2)}\Big\}$$

$$S_{-\exp}(v) = (n-1)\Big\{K_0\Big(n - \frac{K_0}{2}|v|^2 e^{-|v|^2}\Big) + \frac{e^{|v|^2}}{1+|v|^2}\Big[(n-2)(3-|v|^2) + \frac{6+2|v|^2}{1+|v|^2}\Big]\Big\}$$

Corollary 4.17 *Let* (M, g) *be a flat manifold, then we have that:*

a) $S_{+\exp} < 0$.

b) If $\dim M = 2$, *then* $S_{-\exp} > 0$.

c) If $\dim \geq 3$, $S_{\exp}(v) > 0$ *if and only if* $0 \leq |v|^2 < \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}$.

d) Si $\dim \geq 3$, $S_{\exp}(v) = 0$ *if and only if* $|v|^2 = \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}$.

Proof. It follows from Proposition 4.16.

□

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